

## Non-perturbative renormalization in lattice QCD

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Recent developments in non-perturbative renormalization for lattice QCD are reviewed with a particular emphasis on RI/MOM scheme and its variants, RI/SMOM schemes. Summary of recent developments in Schrödinger functional scheme, as well as the summary of related topics are presented. Comparison of strong coupling constant and the strange quark mass from various methods are made.

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## 1. Introduction

In many applications of lattice QCD it is necessary to deal with renormalization. A simplest example is to obtain the quark masses which are the fundamental parameters in the standard model. A bare quark mass is logarithmically divergent in the continuum limit, hence, is not defined without proper regularization and renormalization. A less simple example is to calculate a transition or decay amplitude of hadrons induced by interactions in the electroweak theory. The low energy effective operators are constructed through operator product expansion. They, in turn, are plugged in the QCD Lagrangian, with the Wilson coefficients calculated through (continuum) perturbation theory. The coefficients are often divergent, thus should be defined through renormalization with a certain renormalization scheme and scale. The lattice part is to calculate the matrix element of the operator between hadron states. The matrix element of the bare operator has the same divergence as the Wilson coefficient. In divergent cases, the matrix element or the operator must undergo the renormalization on the lattice. The same renormalization scheme and scale as the Wilson coefficient must be used to compensate the dependence on them, so that the final, physical matrix element is scheme and scale independent.

As these typical examples show, there are two necessary steps required for the lattice renormalization: 1) removing the ultraviolet divergence in the observables and 2) matching to the scheme convenient to the continuum renormalization, such as  $\overline{\text{MS}}$ . The step 2) can be performed in continuum perturbation theory (cPT) once 1) is defined in a regularization independent (RI) way. The lattice perturbation theory (LPT) can be used to perform 1). However, the convergence of the series in general is not good due to tadpole contributions. Although a cure is provided by mean field improved perturbation theory, one has to deal with the ambiguity due to the choice of mean field coupling. In addition, for some quantities the matching 2) has been calculated to three loops in cPT. Such a calculation cannot be fully exploited if the LPT is used for the step 1) due to the difficulty of calculating two or more loops in LPT. These problems can be bypassed by the use of non-perturbative renormalization (NPR) applied for the step 1).

The NPR on the lattice was born early in the previous decade. There have been dedicated plenary presentations in this series of conferences [1, 2, 3, 4].

The popular NPR schemes to date are the momentum subtraction (RI/MOM) and Schrödinger functional (SF) schemes. The SF scheme often refers to its combined use with step scaling. The RI/MOM scheme is widely used in the light quark sector of state-of-the-art lattice calculations today. The use of SF scheme for light quarks is slightly restricted, presumably due to the need of tailored gauge configurations and subtleties when applying to the chiral fermions. The SF scheme has been extended to incorporate the heavy quark effective theory (HQET) on the lattice. It has been discussed in [4] and more recently to some detail by Della Morte [5].

As is well known there is the so called “window problem” in the RI/MOM scheme. To have minimal effect from the problem some practical prescription is required for the RI/MOM scheme. As an extension of the RI/MOM scheme, the RI/SMOM scheme was proposed recently and applied to some realistic calculations. The new scheme is meant to ease the “window problem” by squeezing out unwanted non-perturbative effects from the setup of the scheme. This review focuses mainly on the recent development in RI/MOM and RI/SMOM schemes on how to work around the problem.

In what follows, RI/MOM and RI/SMOM schemes are discussed in Sec. 2. Recent developments in SF scheme are given in Sec. 3. Sec. 4 contains application specific developments, such as the calculation of the strong coupling constant and quark masses. Concluding remarks are given in Sec. 5.

## 2. Developments in the RI/MOM scheme

The RI/MOM scheme [6, 7] is a popular non-perturbative renormalization scheme for multi-quark operators today. It has been applied to virtually all types of light quark discretizations used to date. The renormalization condition is imposed on the forward vertex function of the operator with external off-shell quark states having momentum  $p$ . This scheme is handy as no special gauge configurations are required. A drawback is that there is the so called window problem, i.e. the renormalization scale  $\mu = \sqrt{p^2}$  needs to be in the range  $\Lambda_{QCD} \ll \mu \ll 1/a$ . Finer lattice simulations are less problematic. However for the typical lattice cutoff used today, this could be a source of a sizable systematic error. In principle, the RI/MOM scheme can be used to determine the coefficient of the higher dimensional operators that appear in the Symanzik on-shell improvement. However, more operators have to be introduced than the SF scheme to properly disentangle the linear relation due to the use of off-shell external states, which would make the tuning difficult in practice. Operators with chiral or twisted mass fermions have no problem with this point since there are no  $O(a)$  operators to mix at on-shell for typical applications.

Several remarkable developments in the RI/MOM scheme and its extension are reported by this year. To discuss them it is required to have a review of the RI/MOM scheme.

### 2.1 RI/MOM formulation

For illustrative purpose let us go through the RI/MOM renormalization of a flavor non-singlet quark bilinear operator  $O_\Gamma = \bar{u}\Gamma d$ . The concept is easily extended to three, four  $\dots$  quark operators. Later on, the pseudoscalar/scalar operator is examined in detail for the quark mass renormalization. The bare operator  $O_\Gamma$  with the Dirac structure  $\Gamma$  is renormalized with  $Z_\Gamma$  as

$$O_\Gamma^R = Z_\Gamma O_\Gamma. \quad (2.1)$$

The RI/MOM renormalization condition is imposed to the amputated vertex function  $\Pi_\Gamma(p_1, p_2)$  with Landau-gauge fixed external mass-less off-shell quark states with the same incoming and out-going momenta  $p_1 = p_2 = p$ ,

$$\frac{Z_\Gamma}{Z_q} \text{Tr}[\Pi_\Gamma \cdot \mathcal{P}_\Gamma] = 1, \quad (2.2)$$

where  $Z_q$  is the quark wavefunction renormalization factor, i.e.  $S^R = Z_q S$  with  $S^R$  and  $S$  being the renormalized and bare quark propagator. The projection operator  $\mathcal{P}_\Gamma$  is chosen so that the condition properly removes all the ultraviolet divergence and holds trivially at tree level with  $Z_\Gamma = Z_q = 1$ . The renormalization scale is set through the external off-shell momentum  $\mu^2 = p^2$ .

There are two different schemes due to slightly different definitions of the wavefunction renormalization. The RI/MOM  $Z_q$  is defined through the vector vertex with  $\mathcal{P}_V \propto \gamma_\mu$

$$\frac{Z_V}{Z_q} \frac{1}{48} \text{Tr}[\Pi_{V_\mu} \cdot \gamma_\mu] = 1. \quad (2.3)$$

In the continuum or by the use of conserved vector current on the lattice, since  $Z_V = 1$  this determines  $Z_q$ . With the local vector current on the lattice  $Z_V$  needs to be determined by other means. In the continuum, this condition is equivalent through the Ward-Takahashi (WT) identity with the condition on the inverse quark propagator,

$$\frac{1}{Z_q} \frac{1}{12} \text{Tr} \left[ -i \frac{\partial}{\partial \not{p}} S^{-1}(p) \right] \Big|_{p^2=\mu^2} = 1. \quad (2.4)$$

Another scheme used to date is RI'/MOM scheme where instead of Eq. (2.3) or (2.4)  $Z_q$  is defined through

$$\frac{1}{Z_q} \frac{1}{12} \text{Tr} \left[ -i \frac{\not{p}}{p^2} S^{-1}(p) \right] \Big|_{p^2=\mu^2} = 1. \quad (2.5)$$

These two definitions of  $Z_q$  are quite similar, which can be seen by the fact that the difference only appears at two loops in continuum perturbation theory [8]. To date RI'/MOM scheme is being used mostly for Wilson type fermions, while the RI/MOM is being utilized in domain-wall (DW) and overlap fermions.

Once the  $Z_F$  in RI/MOM or RI'/MOM scheme is obtained, it can be converted to that in other schemes by multiplying a conversion factor calculated in perturbation theory.

These are the essence of RI/MOM schemes. In the lattice computation, there emerges a practical issues associated with the use of momentum for the renormalization scale. The so-called window problem arises as the renormalization scale needs to satisfy two conditions. The first condition is

- $\mu \gg \Lambda_{\text{QCD}}$ .

The matching factor / Wilson coefficient is calculated in perturbation theory. At a given order, the truncation error grows as  $\mu$  decreases. Furthermore, if the operator couples to the Nambu-Goldstone (NG) boson due to spontaneous symmetry breaking (SSB), the effect cannot be corrected by perturbation theory and fails the matching. The condition is required to have these effects small. It should be noted, however, for some operators the SSB effect diverges as  $1/m$  which has to be subtracted. Secondly,

- $\mu \ll 1/a$ .

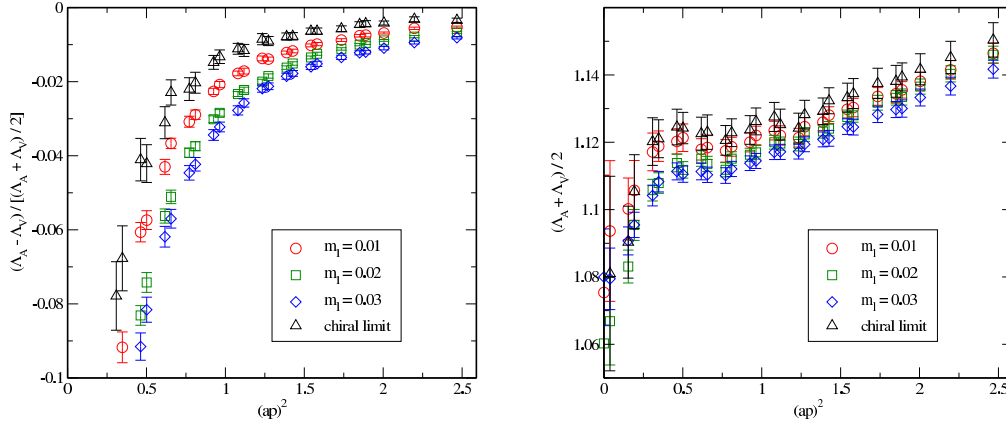
This is to reduce the discretization error, such like  $(pa)^2$ . However, the  $(pa)^2 \geq 1$  region is used in typical calculations. It appears, in practice,  $\mu \ll \pi/a$  is OK.

These conditions provide the guideline to have the systematic error small for the RI/MOM schemes and the matching. A more detailed discussion follows by using an example of quark mass renormalization for the estimate of the systematic errors with  $n_f = 2+1$  DWFs. Here for simplicity we assume exact chiral symmetry. An issue of the small explicit chiral symmetry breaking will be discussed in Sec. 2.4. The mass renormalization for  $m^R = Z_m m$  is conveniently calculated through the relation

$$Z_m = 1/Z_S = 1/Z_P^1, \quad (2.6)$$

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<sup>1</sup> $Z_m = 1/Z_P \neq 1/Z_S$  for Wilson type fermions.



**Figure 1:** Fractional difference (left panel) and average (right panel) of vector and axialvector vertex amplitudes as functions of  $(pa)^2$  for  $n_f = 2 + 1$  DWFs at  $a^{-1} \simeq 1.7$  GeV [9]. Three masses are used to make an extrapolation to chiral limit.

where  $Z_m$ ,  $Z_S$ ,  $Z_P$  are quark mass, flavor non-singlet scalar and pseudoscalar renormalization factors. The ratio  $Z_q/Z_\Gamma$  is calculated from the traced projected vertex functions through the renormalization condition Eq. (2.2),

$$\Lambda_\Gamma = \text{Tr}[\Pi_\Gamma \cdot \mathcal{P}_\Gamma] \rightarrow \frac{Z_q}{Z_\Gamma}. \quad (2.7)$$

It immediately turns out that  $\Lambda_S \neq \Lambda_P$  due to SSB, i.e.,  $\Lambda_P$  diverges as  $1/m$  [9, 10]. Therefore one needs to subtract the divergence or abandon the use of  $\Lambda_P$ . Here we proceed to take the latter choice and use <sup>2</sup>

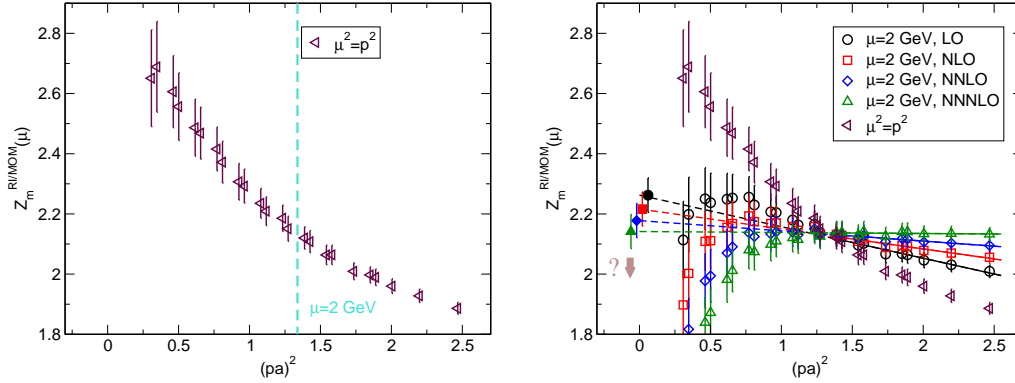
$$\Lambda_S = \frac{1}{12} \text{Tr}[\Pi_S \cdot I] \rightarrow \frac{Z_q}{Z_S} = Z_q Z_m. \quad (2.8)$$

For Wilson type fermions, the pseudoscalar vertex must be used for the mass renormalization. The divergent pion pole contribution must be subtracted [12]. Fitting the quark mass dependence with NLO OPE expression is one way. Or one can construct an object free from the pole combining the multi mass points [13].

The scalar vertex, used in this example, does not have an infrared divergence. However, the naive unitary chiral extrapolation cannot remove a non-perturbative (NP) effect. If one has a handle on the direction of partially quenched mass, the leading NP effect can be isolated [9]. Or one can study the response of the vertex function to the mass to evaluate its sensitivity to the low scale. The latter leads to an estimate of 7% systematic error for the present DWF case.

The quark wave function renormalization is calculated from  $\Lambda_V$  through Eq. (2.3). With a chiral fermion formulation it is often averaged with the traced axial vector vertex  $\Lambda_A$  to gain statistics. In perturbation theory  $\Lambda_V = \Lambda_A$ , but this can be violated at low momenta. The WT identity, i.e. the equivalence of Eqs. (2.3) and (2.4), holds non-perturbatively for the vector current. As the axial current couples with the NG boson, the equality is violated non-perturbatively. Fig. 1 shows an example of the difference of  $\Lambda_A - \Lambda_V$  as a function of  $(pa)^2$ . The difference gives an

<sup>2</sup>In quenched approximation there also be a divergence in the scalar vertex as  $1/m^2$  [11], which is suppressed in dynamical calculation from the quark determinant effect.



**Figure 2:** Left panel shows  $Z_m^{\text{RI/MOM}}(\mu)$  with  $\mu^2 = (pa)^2$ , where cyan line indicates  $p^2 = (2\text{GeV})^2$ . Right panel shows same data and those with RG running stripped off to get  $Z_m^{\text{RI/MOM}}(2\text{GeV})$  by LO to NNNLO PT as a function of matching scale.

estimate of the NP contamination error. We will later use the window  $1.3 < (pa)^2 < 2.5$  where the largest difference is about 1 % which is small compared to the other systematic errors, such as the aforementioned 7%. The right panel shows the average  $(\Lambda_A + \Lambda_V)/2$ . Now the quark mass renormalization is calculated through

$$Z_m^{\text{RI/MOM}}(\mu) = Z_A \frac{2\Lambda_S}{\Lambda_A + \Lambda_V} \Big|_{\mu^2=p^2}, \quad (2.9)$$

where  $Z_A (= Z_V)$  is obtained from a ratio of hadronic two point correlation functions of partially conserved and local axial vector current [9],  $Z_A = \langle \mathcal{A}_0(x)P(0) \rangle / \langle A_0(x)P(0) \rangle$ .

The left panel of Fig. 2 shows  $Z_m^{\text{RI/MOM}}(\mu)$  with  $\mu^2 = p^2$  as a function of  $(pa)^2$ .  $Z_m^{\text{RI/MOM}}(2\text{GeV})$ , for example, can be read off from the intercept with the cyan line drawn at  $p^2 = (2\text{GeV})^2$ . This naive determination, however, might suffer from discretization errors, as well as the NP contamination error which has already been estimated.

For the discretization error a common procedure is to strip off the renormalization-group (RG) running with perturbation theory. If the truncation error is negligible in a certain range of momenta, a linear extrapolation to  $(pa)^2 = 0$  from the range will remove a leading  $(pa)^2$  error. However, as a comparison between different orders of perturbation theory indicates, this extrapolation does not always give a better estimate than the simple method from the intercept (left). A particular problem of the extrapolation is that the remnant truncation error of the perturbative running gets enhanced by  $(pa)^2 \rightarrow 0$ . Thus, if there is any indication of sizable PT truncation error, as is indicated in the figure,  $(pa)^2 \rightarrow 0$  should not be performed.

Fixing the matching scale  $p$  to a non-zero value could result in a biased estimate of the renormalization factor due to a discretization error  $O(p^2 a^2)$  when only one lattice spacing is available. However, keeping the scale in physical unit for instance  $p = 2 \text{ GeV}$  while taking the continuum limit of the renormalized quantity will take away the leading  $(pa)^2$  error automatically. This is clearly better than taking the  $(pa)^2 \rightarrow 0$  limit at fixed  $a$  when a sizable truncation error of PT is suspected.

As the  $(pa)^2$  dependence for  $Z_m$  in the figure is flat with the NNNLO running, it does not matter whether to take the  $(pa)^2 \rightarrow 0$  or not in this particular case. The  $(pa)^2 \rightarrow 0$  value was adopted for  $Z_m^{\text{RI/MOM}}(2\text{GeV})$  in [9].  $Z_m^{\overline{\text{MS}}}(2\text{GeV})$  is obtained by multiplying the conversion factor calculated to three loops [8], Eq. (2.12). The truncation error is estimated from the size of the three-loop term, which turns out to be 6%. The resulting renormalization factor is  $Z_m^{\overline{\text{MS}}}(2\text{GeV}) = 1.656(157)$ , where all the errors (mass dependence 7%,  $\Lambda_A - \Lambda_V$  1%, PT truncation error 6%) have been summed up in quadrature with the statistical error.

## 2.2 RI/MOM new developments

As described above there are three competing effects for possible systematic uncertainty in the RI/MOM schemes which are:

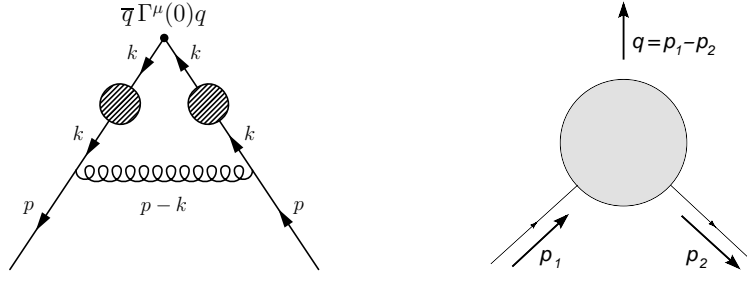
1. unwanted non-perturbative effects,
2.  $(pa)^2$  and higher,
3.  $\alpha_s^{n+1}(\mu)$  and higher.

The last error is relevant for the renormalization factor in  $\overline{\text{MS}}$ , or in the case where the perturbative running are used for whatever purpose. There are several ideas to overcome one of them or aiming to reduce most of them. Here the methods are explained for the recent large scale calculations for the quark mass renormalization by four collaborations: ETM for  $n_f = 2$  twisted mass fermions with RI'/MOM [14, 15], QCDSF/UKQCD for  $n_f = 2$  improved Wilson fermions with RI'/MOM [16, 17], JLQCD/TWQCD for  $n_f = 2$  and  $2 + 1$  overlap fermions with RI/MOM [18], and RBC/UKQCD for  $n_f = 2 + 1$  DWFs with RI/SMOM and RI/SMOM $_{\gamma_\mu}$  [19].

For 1, ETM, QCDSF/UKQCD and JLQCD/TWQCD perform the subtraction. ETM and JLQCD/TWQCD exploit the partially quenched data for the fit to subtract the leading effects. JLQCD/TWQCD also performed the RI/MOM analysis on the  $\varepsilon$ -regime where the SSB effect is suppressed. They demonstrated that  $\Lambda_A - \Lambda_V = 0$  holds well in the region of  $p^2$  where even a few percent difference was observed with p-regime data. RBC/UKQCD uses SMOM schemes (see Sec. 2.3).

For 2, ETM performs  $O(g^2 a^2)$  subtraction for terms like  $(pa)^2$  and  $\sum_\mu (p_\mu^4/p^2)a^2$ , obtained through analytic calculation in lattice perturbation theory [20]. QCDSF/UKQCD uses a similar subtraction, but, with a numerical integration in lattice perturbation theory. The two collaborations extrapolate remnant  $(pa)^2 \rightarrow 0$ . RBC/UKQCD fixes  $p = 2$  GeV, expecting the continuum extrapolation of the renormalized quantity with  $a^2 \rightarrow 0$  removes the  $(pa)^2$  error. As a second method ETM performs the similar procedure as RBC/UKQCD and they checked the consistency with the first method after the continuum extrapolation.

The RI/MOM to  $\overline{\text{MS}}$  conversion factor for the quark mass is available to  $\alpha_s^3$  [8]. JLQCD/TWQCD estimates the size of the  $\alpha_s^4$  contribution from the extrapolation of  $\alpha_s^n$  contribution for  $n \leq 3$  and take it as the truncation error. RBC/UKQCD study the difference of  $Z_m^{\overline{\text{MS}}}$  from two different RI/SMOM intermediate schemes for the truncation error of RI/SMOM  $\rightarrow \overline{\text{MS}}$  perturbative conversion. They also check the consistency of the error with the size of the highest order in the conversion factor.



**Figure 3:** A sample figure which makes  $1/p^2$  contribution with SSB from Ref. [9] (left) and general momentum configuration of bilinear vertex (right) including the exceptional ( $q = 0$ ) and symmetric non-exceptional ( $p_1^2 = p_2^2 = q^2$ ) momenta from Ref. [22].

### 2.3 RI/SMOM schemes

It was demonstrated in Ref. [9] that the severe infrared sensitivity of the RI/MOM scheme can significantly be reduced by avoiding the use of exceptional momenta. The argument stems from Weinberg’s theorem [21] on the behavior of the vertex function for large external momenta, where a set of external momenta which has zero partial sum is called exceptional.

In the case of bilinear operators with exceptional momenta, the momentum that flows in from one fermion leg can be rerouted to the other leg by one-gluon exchange (see Fig. 3). If this happens the operator part allows a small momentum flow, which triggers SSB. This contamination only suppresses as  $1/p^2$  at high momenta, hence could be sizable for the typical allowed momentum in simulations. This can be avoided by eliminating the exceptional channel by setting all  $p_1$ ,  $p_2$  and  $q$  non-zero in the right panel in Fig. 3.

The RI/SMOM and  $\text{RI/SMOM}_{\gamma_\mu}$  schemes are constructed for the quark bilinear operators in Ref. [22], exploiting the non-exceptional momenta and carefully ensuring the chiral Ward-Takahashi identities are satisfied. The symmetric configuration  $p_1^2 = p_2^2 = q^2$  is used, hence, “symmetric” MOM scheme. A trial calculation [10] showed the RI/SMOM scheme is a promising alternative to the conventional RI/MOM scheme with reduced systematic uncertainties.

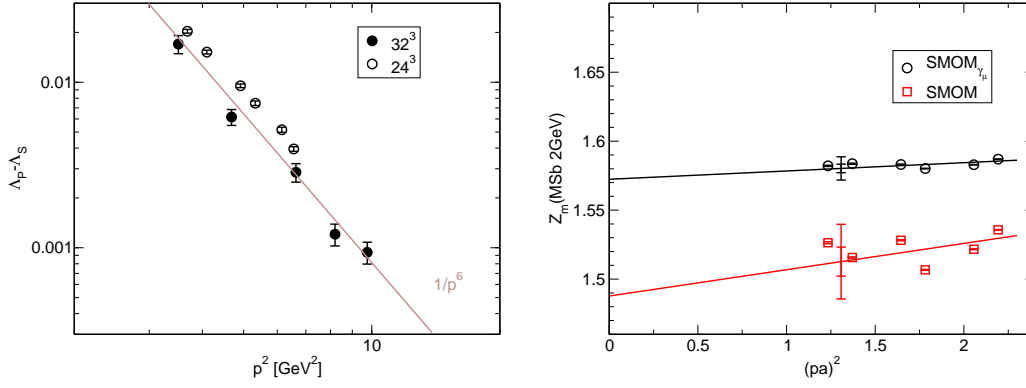
The RI/SMOM scheme uses a different projector  $\mathcal{P} \propto \not{q} q_\mu / q^2$  compared to  $\mathcal{P} \propto \gamma_\mu$  in Eq. (2.3). The resulting quark wavefunction renormalization is equivalent to that of the RI’/MOM scheme, i.e.  $Z_q^{\text{RI/SMOM}} = Z_q^{\text{RI’/MOM}}$ . Another scheme named  $\text{RI/SMOM}_{\gamma_\mu}$  uses the same projector as the RI/MOM scheme  $\mathcal{P} \propto \gamma_\mu$ . It should be remembered, though, due to the use of the symmetric momentum, the resulting wave function renormalization is different, i.e.  $Z_q^{\text{RI/SMOM}_{\gamma_\mu}} \neq Z_q^{\text{RI/MOM}}$ . Not only the reduction of the unwanted non-perturbative effect, but the better convergence of the PT for the matching to  $\overline{\text{MS}}$  from SMOM schemes are observed. The matching was calculated to one-loop [22]. Evaluating the magnitude at each order in  $\alpha_s$ <sup>3</sup>,

$$C_m(\text{RI/SMOM} \rightarrow \overline{\text{MS}}, \mu = 2\text{GeV}, n_f = 3) = 1 - 0.015 + \dots, \quad (2.10)$$

$$C_m(\text{RI/SMOM}_{\gamma_\mu} \rightarrow \overline{\text{MS}}, \mu = 2\text{GeV}, n_f = 3) = 1 - 0.045 + \dots. \quad (2.11)$$

<sup>3</sup>  $\alpha_s^{\overline{\text{MS}}(3)}(2\text{GeV}) = 0.2907$  from four loop beta function has been used.





**Figure 4:** The difference  $\Lambda_S - \Lambda_P$  as a function of  $p^2$  GeV<sup>2</sup> for  $a \simeq 0.11$  ( $24^3$ ) and  $0.09$  fm ( $32^3$ ) DWF lattices (left) [19]. Solid straight line is proportional  $1/p^6$  for guide for eyes. Right panel shows  $Z_m^{\overline{\text{MS}}}(2\text{GeV})$  with the SMOM or SMOM $_{\gamma_\mu}$  intermediate NPR scheme as function of  $(pa)^2$  for the same  $a \simeq 0.11$  fm DWF lattice.

In the RI/MOM and RI'/MOM schemes the conversion factors are known to three-loop order [8, 23]:

$$C_m(\text{RI/MOM} \rightarrow \overline{\text{MS}}, \mu = 2\text{GeV}, n_f = 3) = 1 - 0.123 - 0.070 - 0.048 + \dots, \quad (2.12)$$

$$C_m(\text{RI'/MOM} \rightarrow \overline{\text{MS}}, \mu = 2\text{GeV}, n_f = 3) = 1 - 0.123 - 0.065 - 0.044 + \dots. \quad (2.13)$$

The convergence of the new SMOM schemes to  $\overline{\text{MS}}$  is better. Moreover, it has recently been confirmed that this trend continues to two loops [24, 25], strongly suggesting the truncation error of the matching is significantly small for the SMOM schemes.

The new SMOM schemes are being used for the quark mass renormalization in  $n_f = 2 + 1$  DWFs with two lattice spacings by RBC/UKQCD collaborations [19]. The left panel of Fig. 4 shows the difference of pseudoscalar and scalar vertex amplitude  $\Lambda_P - \Lambda_S$ , as an indicator of the unwanted non-perturbative effect, at the chiral limit. Not only is the difference finite, but small (1% at  $p = 2$  GeV for instance), which is in clear contrast to the same quantity in the MOM scheme where it was divergent in the chiral limit as  $1/(mp^2)$ . Furthermore it exhibits the  $1/p^6$  behaviour expected from Weinberg's power counting and the possible symmetry breaking pattern for the non-exceptional momenta [9].

The right panel shows the  $Z_m^{\overline{\text{MS}}}(2\text{GeV})$  for the  $a \simeq 0.11$  fm lattice, evaluated with the SMOM or SMOM $_{\gamma_\mu}$  intermediate NPR scheme combined with one-loop matching [22] and two-loop  $\overline{\text{MS}}$  running as function of the matching scale squared. The origin of the inconsistency could be the truncation error of the perturbation theory and the lattice discretization error. The non-perturbative effect is too small to explain this difference. The existence of the two independent estimates of  $Z_m^{\overline{\text{MS}}}$  helps to estimate the systematic error. A preliminary estimate of the central value from an error weighted average at  $p = 2$  GeV, taking into account the spread of the point around the linear fit, a variation with respect to the change of the matching scale is given as  $Z_m^{\overline{\text{MS}}}(2\text{GeV}) = 1.568(75)$ , where the error includes an estimate of PT truncation error. This value is consistent with the same quantity obtained with the conventional RI/MOM scheme,  $Z_m^{\overline{\text{MS}}}(2\text{GeV}) = 1.656(157)$  [9], but with

a considerably reduced error. Further error reduction is expected by incorporating the two-loop matching factors.

The non-exceptional momentum configuration has been applied to the renormalization of non standard-model  $B_K$  operators, where the usual RI/MOM kinematics produces  $1/(mp^2)$  type infrared divergence. The non-exceptional momenta removed these divergences completely as expected [19].

An extension of SMOM kinematics to the standard-model four Fermi operator for  $B_K$  uses SMOM and  $\text{SMOM}_{\gamma_\mu} Z_q$ . For the  $\Gamma$  projector for the four quark operator, one can take the conventional type as well as the one with momentum transfer  $q_\mu$  in analogy to the SMOM and  $\text{SMOM}_{\gamma_\mu}$  projector. The  $2 \times 2$  combinations make four SMOM schemes. Including the original RI/MOM scheme, five independent schemes lead to a better estimate of the central value of  $Z_{B_K}^{\text{NDR}}(2\text{GeV})$  and the systematic error [19].

The advantages and remaining issues of the SMOM schemes are summarized as follows. Some have already been mentioned. The RI/SMOM schemes improved from the RI/MOM scheme on:

- Contamination of the unwanted non-perturbative SSB effect is greatly reduced.
- Mass dependence is very weak.
- PT matching to  $\overline{\text{MS}}$  has far better convergence.
- Signal/noise ratio for the non-exceptional momentum is improved from the exceptional.

The second point is consistent with the first point and particularly convenient for  $n_f = 2 + 1$  calculations, where one can safely keep the strange mass around the physical point. The last point is likely due to the insensitivity of non-exceptional vertex functions to the low energy fluctuation. There could be a room for an improvement on

- enhanced effect of  $O(4)$  breaking in momentum space.

The last issue is visible in the right panel of Fig. 4 especially for the RI/SMOM scheme. It manifests itself as a non-smooth behaviour in  $(pa)^2$ . Such an effect persists also in the MOM scheme, but smaller. The leading  $O(4)$  breaking term appears at  $O(g^2 a^2)$  with a structure of  $\sum_\mu p_\mu^4 / p^2$  [20] for MOM scheme. As there is no precise knowledge of such terms in SMOM schemes, the effect has just been estimated from the spread of the data and taken into account in the systematic error. Given that now the two-loop conversion factor is available, this error could dominate the systematic error in the near future, then the solution to this issue will be desired.

## 2.4 Other remarks in RI/MOM schemes

For DWFs at finite fifth dimension  $L_5$  the axial Ward-Takahashi identity has an additional contribution to the naive extension from the continuum one from the pseudoscalar like the operator at mid point of fifth dimension. As such the partially conserved axial current  $\mathcal{A}_\mu^a$  does not conserve in the chiral limit, which makes the renormalization factor of the current deviate from 1 [26]. Denoting the additive renormalization of the quark mass in lattice units due to the non-vanishing midpoint term as  $m_{\text{res}}$ , the effect was estimated as  $Z_{\mathcal{A}} = 1 + O(m_{\text{res}})$  for the perturbative and  $Z_{\mathcal{A}} = 1 + O(m_{\text{res}}^2)$  for the non-perturbative contribution [27]. The size of the correction was estimated to

be no larger than 1% for  $n_f = 2 + 1$  DWFs at  $a \simeq 0.11$  fm. As the ratio of the local and partially conserved axial current is an input to the RI/MOM schemes to evaluate the quark wavefunction renormalization, this systematic error will propagate to the renormalization factors for multi-quark operators and the quark mass. This is not a dominant error in the current calculations. However, if the other errors are getting smaller, eventually this issue will need to be revisited.

It is reported in Ref. [9] that the use of point sources for the RI/MOM scheme can result in underestimating the statistical error. The problem is enhanced towards the larger momenta. This is understandable because a larger momentum in the Fourier transformation of the sink position tends to sample a narrower region in the real space, thus effectively reduces the statistics. One solution is to have several distant source positions or to have a random source position. Another is the use of a volume momentum source, which effectively samples the entire volume.

### 3. Developments in Schrödinger functional scheme

Finite volume scheme with Schrödinger functional (SF) boundary conditions combined with a step scaling analysis provides a solid framework to perform a precision non-perturbative renormalization [28]. The renormalization group running of the operators in the continuum theory can be calculated in a very efficient manner. Possible applications include the determination of the running coupling constant, renormalization of multi-quark operators, coefficients of Symanzik improvement operators. The operator renormalization condition is applied to correlation functions of the operator in the bulk and distant boundary field(s). As such, the scheme is compatible for the Symanzik on-shell improvement of the operators. The practical difficulty is that the scheme needs tailored gauge ensemble generation for many parameter points for the step scaling and continuum limit. However, once the running of the particular operator is determined (with any gauge-fermion action), the remaining process is to perform the matching at given lattice spacing.

The technique has been explained to some detail in the plenary presentations in the past conferences (see [3, 4]). In the plenary talk of Lattice 2002 by Sommer, who gave the last plenary presentation dedicated on the non-perturbative renormalization until this year, he listed three problems which remain to be tackled. All related to the SF scheme:

- Development of SF scheme for HQET at  $1/m_h$ ,
- $n_f = 3$  simulation for non-perturbative estimate of  $\alpha_{\text{SF}}(\mu)$ ,
- applicability of SF scheme for the chiral fermions.

By this year the answers of these questions have been given. The non-perturbative renormalization of HQET formulation at  $1/m_h$  is important to b-quark physics. In a recent lattice conference the calculation strategy has been explained in detail by Della Morte [5]. This year the Alpha collaboration reported on the calculation of ground state and first excited state spectrum of bottom-strange mesons, as well as the decay constant of  $B_s$ , with completing the continuum extrapolation [29]. Application to  $n_f = 2$  improved Wilson fermion is underway.

$B^0 - \bar{B}^0$  mixing matrix elements have great importance to flavor physics (for example to constrain the apex of the unitarity triangle of CKM matrix [30, 31]). So far the renormalized operators are constructed in the static approximation to b-quark for  $n_f = 2$  [32] and  $n_f = 2$  [33]. If the

$O(1/m_h)$  operator is constructed, the systematic error from  $O(1/m_h^2)$  could be constrained to sub-percent level. However, it is yet to be realized.

CP-PACS and PACS-CS collaborations have been performing  $n_f = 2 + 1$  simulations with non-perturbatively improved Wilson fermions. SF coupling  $\alpha_{\text{SF}}$  is obtained by step scaling to give a high accuracy estimate of  $n_f = 3$  coupling at a high scale, where matching to  $\overline{\text{MS}}$  scheme is performed. Running down to charm threshold  $\mu = m_c$  by perturbative running to match to  $n_f = 4$  and then running up to  $n_f = 5$ ,  $\mu = m_Z$ , they obtained  $\alpha_s(m_Z)$  [34]. See Sec. 4.1 for comparison with the other estimate. They push the project further to calculate the quark mass renormalization through SF scheme. The preliminary results are reported in [35]. The resulting renormalized quark mass is discussed in Sec. 4.2 together with those from other discretization/NPR schemes.

The SF boundary condition on the quark propagator naively interferes with the Ginsparg-Wilson (GW) relation at the boundary. As such, the GW relation has to be modified to properly define the SF formalism on overlap or domain-wall fermions (DWFs). The solution has been provided by Taniguchi for quenched overlap [36] and DW [37] fermions through orbifolding. This method is successfully applied to the renormalization for the light quark mass and  $B_K$  with DWFs in the quenched approximation [38]. It is extended by Sint [39] for the even number of flavors case, and Lüscher [40] for any number of flavors for overlap fermions. This year further development has been reported by Takeda [41] on SF scheme construction in DWF with any number of fermions.

The SF scheme is being used for twisted mass simulations as well. ETM collaboration tune the boundary counter term of “chirally rotated SF” for twisted mass fermions non-perturbatively [42] to ensure the automatic  $O(a)$  improvement. The tuning has been performed for the quenched approximation. Application to  $n_f = 2$  is underway.

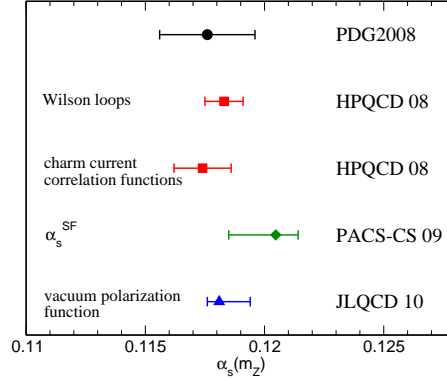
## 4. Application specific developments/remarks

In this section application specific developments in recent years and some comparison of different schemes are provided.

### 4.1 Strong coupling constant $\alpha_s$

Due to the availability of several independent  $n_f = 2 + 1$  simulations in recent years, lattice estimates of  $\alpha_s^{\overline{\text{MS}}}(M_Z)$  have now become very strong compared to the other methods. This has not been possible when only quench or  $n_f = 2$  simulations were available, because of the inaccessibility of the threshold scale  $\mu = m_{u,d}$  or  $m_s$  in perturbation theory. An estimate of  $\alpha_s(\mu)$  at  $n_f = 3$  for  $\mu = O(1)$  GeV allows one to use perturbative matching at charm threshold  $\mu = m_c$  by perturbation theory (PT) to  $n_f = 4$ , and then at  $\mu = m_b$  to  $n_f = 5$ , with RG evolution with anomalous dimension calculated with PT. The systematic error which arises in this calculation apart from the lattice non-perturbative part mainly comes from 1) the PT machining of a particular scheme used in NP part to  $\overline{\text{MS}}$  and 2) the  $n_f = 3 \leftrightarrow 4$  matching at  $\mu = m_c$  in  $\overline{\text{MS}}$  scheme. Due to 2), one needs to select a scheme with which the matching is available to high order typically to  $\alpha_s^3$ .

There have been four estimates of  $\alpha_s^{\overline{\text{MS}}}(M_Z)$  by the conference, 1) through  $\alpha_V$  calculated from Wilson loops [47, 44], 2) from moments of the charmonium current correlators [45], 3) from SF coupling [34], 4) from vacuum polarization function [48, 46]. For 1) one calculates Wilson loops to get  $\alpha_V(\mu)$  through lattice perturbation theory, and then matched to  $\alpha_s^{\overline{\text{MS}}}(\mu)$  at three-loop



**Figure 5:** Summary of  $\alpha_s^{\overline{\text{MS}}}(M_Z)$  from  $n_f = 2 + 1$  lattice simulations, compared with PDG 2008 average [43] (black). Results are from Wilson loop [44] and charm correlation functions [45] with staggered (red), SF coupling [34] with Wilson (green), and vacuum polarization function [46] with overlap fermions (blue).

(NNNLO). 2) uses the continuum three-loop expression (NNNLO) of a moment of charm current-current correlation function. This is a by-product of their charm quark mass estimate for which they use same correlation functions. 3) use step scaling of the SF coupling and matching to  $\overline{\text{MS}}$ . Although this has been obtained only to two loops (NNLO), the matching is performed at high scale  $\mu \gg m_c$ , where the PT truncation error is small compared to the other errors. The renormalization scale is set from the inverse linear lattice extension  $\mu = 1/L$ . Once  $\alpha_s^{\overline{\text{MS}}(3)}(\mu)$  has been obtained, it is run down to  $\mu' = m_c$  to match to  $n_f = 4$ . This matching and running are done with NNNLO. 4) the continuum vacuum polarization function has been obtained through operator product expansion and the relevant coefficients has been calculated to NNNLO. The renormalization scale is set from the size of the injected momentum at the current. A compiled summary of  $\alpha_s^{\overline{\text{MS}}}(M_Z)$  estimates is shown in Fig. 5. Results from different method / lattice discretization are consistent with each other and with the PDG2008 average [43].

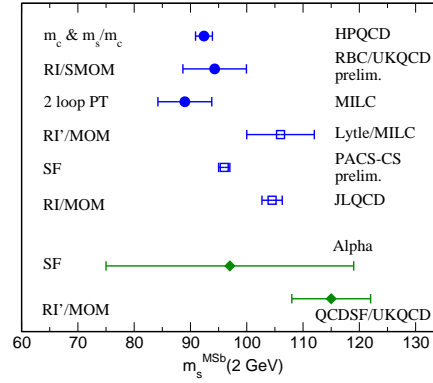
All of these calculations reached a remarkable precision of the strong coupling constant. One has to keep in mind, however, all of them uses the perturbation theory at  $\mu \simeq m_c$ , where non-perturbative effect might still be important. Analysis on  $n_f = 2 + 1 + 1$  simulations on finer lattices will be able to address this issue.

There are several technical development on determining  $\Lambda_{\overline{\text{MS}}}$  using the Landau-gauge ghost and gluon propagators, which has been applied to  $n_f = 0$  and 2 [49, 50, 51].

## 4.2 Light quark mass

A comparison of the light quark mass from different lattice discretization and renormalization schemes is presented by Scholz [60] in this conference. Here for the purpose of discussing the renormalization issues the strange mass compilation is made with the available data in the literature (Fig. 6).

For  $n_f = 2 + 1$  dynamical fermion computations, a remarkable precision has been achieved by HPQCD collaboration [61, 52] who exploits a precise estimate of  $\overline{\text{MS}}$  charm quark mass from the charmonium current correlator and the bare mass ratio  $m_s/m_c$ . The use of highly improved staggered quark (HISQ) and the fine lattice spacing made it possible to use the same action for



**Figure 6:** Summary of strange mass  $m_s^{\overline{\text{MS}}}(2\text{GeV})$  for  $n_f = 2$  and  $2 + 1$  simulations. Blue solid symbols are from  $n_f = 2 + 1$  simulations with continuum extrapolation [52, 53, 54]. Blue open symbols are from  $n_f = 2 + 1$  simulations at single non-zero lattice spacing [55, 56, 57]. Green symbols are from  $n_f = 2$  calculations [58] [59].

both light and charm quarks. This method does not require the conventional lattice renormalization procedure. RBC and UKQCD collaborations [53] have used RI/SMOM schemes described in Sec. 2.3 on DWFs with two lattice spacings for the continuum extrapolation. The MILC collaboration [54] uses Asqtad action with two loop lattice perturbation theory [62] for the renormalization. These three results in the continuum limit agree with each other. One cannot find any issue for the perturbative vs. non-perturbative renormalization here.

For results from single lattice spacing, Lytle reported the strange mass with MILC coarse lattice with RI'/MOM NPR [55]. He compared with the strange mass with perturbative renormalization on the same lattice and found NPR gives about 20% larger value. PACS-CS reported the calculation from improved Wilson fermions directly simulated on the physical  $ud$  and  $s$  point with Schrödinger functional NPR [35, 57], which has become 30% larger from their earlier estimate with perturbative renormalization referred in Scholz's compilation [60]. These differences between the perturbative vs. non-perturbative renormalization are not necessarily inconsistent with the observed consistency for the values after the continuum extrapolation. This is because the difference could arise from the discretization error in different renormalization schemes. JLQCD/TWQCD has the estimate from overlap fermions with RI/MOM NPR. For these three studies, extension to the second lattice spacing are anticipated.

There has been several  $n_f = 2$  calculations reported for the light quark mass. In this figure of  $m_s$  compilation only earlier studies with Wilson fermions by Alpha [58] and QCDSF-UKQCD [59] collaboration are shown. The new calculation of average  $ud$  quark mass in the continuum limit with twisted mass fermions combined with the RI'/MOM scheme has been reported [63, 64]. The result is consistent with  $n_f = 2 + 1$  continuum values [60].

### 4.3 Three quark operators

A class of three quark operators which are a part of baryon-number violating four-fermion operators induced by OPE from grand unified theories are important for the estimate of proton

lifetime. The operator classification, construction of RI/MOM scheme, and one-loop matching to  $\overline{\text{MS}}$  have been carried out in Ref. [65].

The RI/MOM scheme of more general three-quark operators are constructed by QCDSF-UKQCD [66], for the proton decay matrix elements and nucleon distribution amplitude [67].

#### 4.4 Static heavy quark

As mentioned, the SF scheme has been developed for b-quark system namely heavy-light bilinear and  $\Delta F = 2$  four quark operators with the static approximation. Application of the RI/MOM scheme to this system was tried long ago [68]. The subtlety to have a mass independent scheme for the static quark formulation, where the self-energy has power divergence, prevented further development. In this conference, Izubuchi showed his study on the static quark self-energy, hoping to resolve this issue by making use of the odd part in energy of the Fourier transformed static quark propagator. Further investigation/demonstration is desired.

#### 4.5 Relativistic heavy quark

The Fermilab-Tsukuba approach [69, 70] the heavy quark uses relativistic clover type fermions. The  $(m_h a)^n$  error with all order in  $n$  is removed using perturbation theory. The Columbia group refined the strategy of this approach to correctly count the independent operator coefficients [71] and developed the non-perturbative determination with step-scaling [72]. Later another tuning procedure was proposed with by-passing the step scaling and matching experimental spectrum directly. It has been applied to charm [73] and bottom [72] systems.

### 5. Related development in lattice perturbation theory

Lattice perturbation theory has been developed for some quantities. For the quark mass renormalization, two-loop results are available for staggered [62] and Wilson(W)/clover(C) fermions [74]. The latter calculates the renormalization constant in RI'/MOM scheme. Once that is obtained the continuum matching can be used to convert it, for example, to  $\overline{\text{MS}}$ , where the conversion factor has already been calculated up to three loops [8, 23].

$O(g^2 a^2)$  calculations have been performed on the quark self-energy and bilinear operators for W/C [20], twist two bilinears for W/C/twisted mass (tm) [75], four Fermi operators for W/tm fermions [76]. The first one has been used in combination with the non-perturbative RI'/MOM renormalization to have  $O(g^2 a^2)$  effect on  $p^2$  and  $\sum_\mu p_\mu^4/p^2$  subtracted (see Sec. 2.2).

Numerical stochastic perturbation theory (NSPS) [77] is being developed for the improved gauge and Wilson fermion action on the quark self-energy [78] at three loop through the RI'/MOM renormalization condition imposed in the lattice perturbation theory. Two-loop result for the Symanzik improvement coefficient  $c_{\text{SW}}$  and  $c_A$  for Wilson fermions has been obtained [79]. The Landau-gauge ghost propagator has been calculated to three loop [80], anticipating the application to estimate of  $\alpha_s$ .

### 6. Concluding remarks

Today, the  $n_f = 2 + 1$  simulations, incorporating the full dynamics of three lighter quarks that cannot be accessed from the perturbation theory, are available with four different fermion dis-

cretizations. Thus various crosschecks on the calculated quantities can be performed. Furthermore,  $n_f = 2 + 1 + 1$  simulations, incorporating the dynamical charm effect, for two fermion actions are being carried out [81, 82]. In these circumstances the renormalization in lattice QCD has become more important than ever before. Non-perturbative renormalization, which was reviewed here is a powerful tool to tackle the precision calculation of the fundamental parameters of standard model and hadron-operator matrix elements. In recent years, the Schrödinger functional scheme has been applied to the strong coupling constant and light quark masses for  $n_f = 3$ , non-perturbative tuning of the parameters for the simulations, as well as the b-quark systems. Due to the window problem, RI/MOM schemes at the typical lattice spacings of current simulations could suffer from larger systematic errors than the SF scheme. Yet, there has been several ideas of bringing the error under control, which have been discussed in this review. As a promising new direction the RI/SMOM schemes were described in some detail. A small summary was made for the development on the lattice perturbation theory, as well. The comparison of the strong coupling constant and the strange quark mass estimate in the continuum limit have shown consistency despite of the difference of the lattice action and renormalization method used. The success stands upon the cultivation of not only one but multiple relevant technologies in the lattice QCD community over the last two decades, and would motivate further improvement/development in this stimulating field.

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